Lecture 9

Last time: "off-the-shelf trajectory optimization"

Today: terminology of traj. opt. and
powered descent guidance

→ "Introduction to Trajectory Optimization"
by Matthew Kelly [2017].

$$\min_{t_0, t_F, x(t), u(t)} \quad J(t_0, t_F, x(t_0), x(t_F); p(t_F)) + \int_{t_0}^{t_F} w(\tau, x(\tau), u(\tau); p(\tau))\, d\tau$$

$J(t_0, t_F, x(t_0), x(t_F); p(t_F))$: Mayer term (~ terminal cost)

$\int_{t_0}^{t_F} w(\tau, x(\tau), u(\tau); p(\tau))\, d\tau$: Lagrange term (~ stage cost)

colloquially: terminal cost
stage cost

decision variables: what we solve for

$x(t) \in \mathbb{R}^{n_x}$

$u(t) \in \mathbb{R}^{n_u}$

$t_0 \in \mathbb{R}_{++}$ $\quad t_F \in \underbrace{\mathbb{R}_{++}}_{\text{positive number}}$

$p(t) \in \mathbb{R}^{n_p}$: parameters (not decision variables

$\mathcal{P}[p(t)]$

$$\min_{t_0, t_F, x(t), u(t)} \quad J(t_0, t_F, x(t_0), x(t_F); p(t_F)) + \int_{t_0}^{t_F} w(\tau, x(\tau), u(\tau); p(\tau))\, d\tau$$

subj. to : $\dot{x}(t) = f(t, x(t), u(t); p(t)) \quad \forall t \in [t_0, t_F]$ (system dynamics)

$g(t, x(t), u(t); p(t)) \leq 0 \quad \forall t \in [t_0, t_F]$ (path constraints)

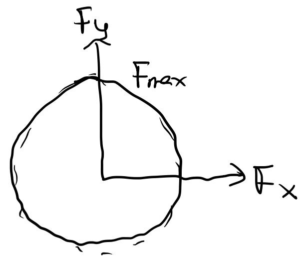
$g(t_0, t_F, x(t_0), x(t_F); p(t_F)) \leq 0$ (boundary conditions)

Common path constraints:

1. $x_{min} \le x(t) \le x_{max} \quad \forall t \in [t_0, t_F]$ (box constraints)

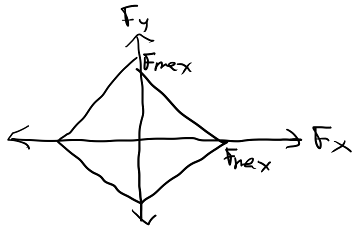
2. $u_{min} \le u(t) \le u_{max} \quad \forall t \in [t_0, t_F]$

3. $\|u(t)\|_2 \le u_{max}$ (ℓ_2-norm constraint)



4. $\|u(t)\|_1 \le u_{max}$ (ℓ_1-norm constraint)

$$\sum_{i=1}^{p} |u_i| \le u_{max}$$



ℓ_1-norm is "sparsity promoting"

Common stage costs:

1. $\int_{T_0}^{T_f} \|u(t)\|_{\ell_1} \, dt$ (minimum fuel)

2. $\int_{T_0}^{T_f} \|u(t)\|_{\ell_2} \, dt$ (minimum energy)

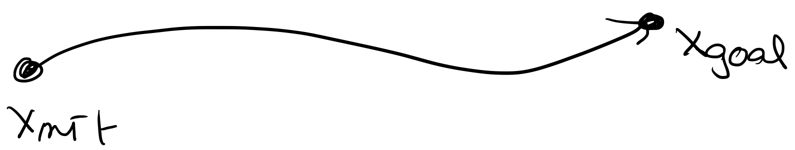
3. $\int_{T_0}^{T_f} dt$ (minimum time)

4. $\int_{T_0}^{T_f} (x(t)-x_g)^T Q (x(t)-x_g) + u(t)^T R u(t) \, dt$ (linear quadratic regulator if $\dot{x} = Ax + Bu$)

Common classes of traj. opt. problems:

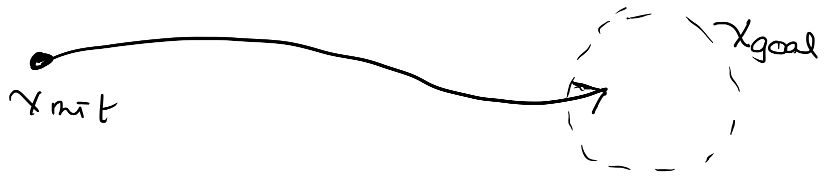
1. Two-point boundary value problem (2PBVP)

$$x(t_0) = x_{init} \qquad x(t_F) = x_{goal}$$



vs. free-final state problem

$$x(t_F) \in \mathcal{X}_{goal}$$



e.g. $\|x(t_F) - x_{goal}\|_{\ell_2} \le r_{dist}$

2. Fixed vs. free final time

Fixed time: t_F is set beforehand

Free final time: t_F is a decision variable

How to convert from infinite-dimensional trajectory optimization to something we can solve?

1. Optimize and then discretize → calculus of variations (might return later in semester)

2. Discretize and optimize: allows us to use "off-the-shelf" nonlinear optimization packages from last time

Approach 2: start with $\mathcal{P}$ above

1. Reformulate $\mathcal{P}$ into a more tractable infinite dimensional form

e.g. if $\mathcal{P}$ is non-convex, convert it to convex form before discretizing

2. Discretize

e.g. Dynamics: $\dot{x}(t) = f(x(t), u(t))$

→ convert to $x_{k+1} = f_k(x_k, u_k)$

e.g. Similary for constraints:

$$x_{min} \leq x(t) \leq x_{max} \Rightarrow x_{min} \leq x_k \leq x_{max}$$

Pitfall: might violate constraint between x_k and x_{k+1}

constraint: $x_k \notin \mathcal{X}_{obstacle}$

e. g. manifold constraints

$$q \in S^3 \quad \|q(t)\|_2 = 1$$

$$\dot{q} = \frac{1}{2} \Xi(\omega) q$$

$$\downarrow$$

$$q_{k+1} = q_k + \dot{q}_k \Delta t$$ (q_{k+1} violates constraint)

Now let's re-write $\mathcal{P}$ as discretized $\bar{\mathcal{P}}$:

$$\bar{\mathcal{P}}[p_{0:N}] \qquad \min_{x_{0:N},\, u_{0:N},\, \Delta t_{0:N}} \underbrace{l_N(x_N)}_{\text{terminal cost}} + \sum_{k=0}^{N-1} \underbrace{l_k(x_k, u_k, \Delta t_k;\, p_k)}_{\text{stage cost}}$$

$\uparrow$

parameters p_k

subj. to: $\quad x_{k+1} = \bar{f}(x_k, u_k, \Delta t_k : p_k) \qquad k = 0, \ldots, N-1$

$$g_i(x_k, u_k, \Delta t_k;\, p_k) \leq 0 \qquad i = 1, \ldots, n_{ineq}, \quad k = 0, \ldots, N-1$$

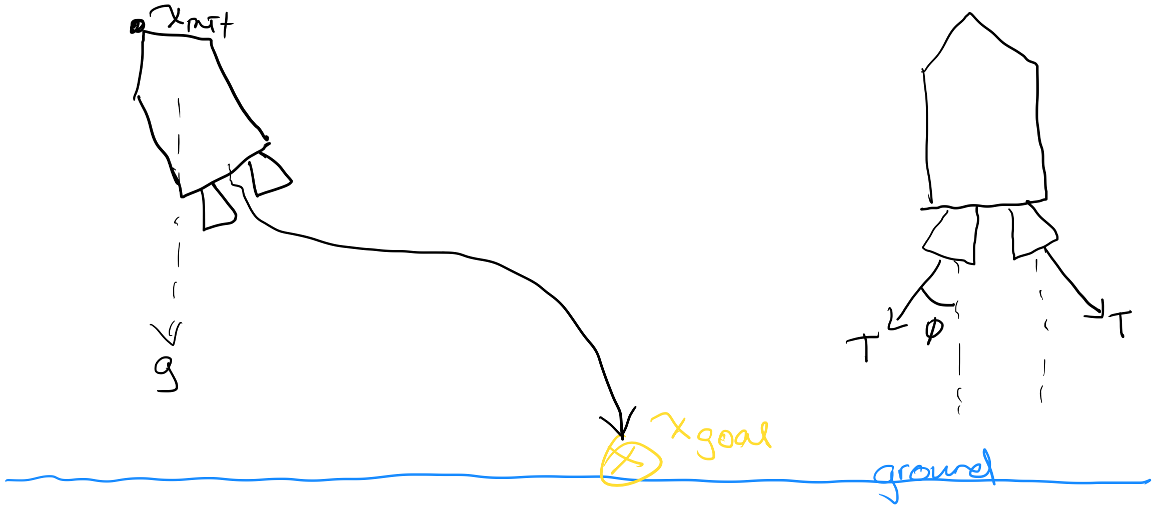
$$h_i(x_k, u_k, \Delta t_k;\, p_k) = 0 \qquad i = 1, \ldots, n_{eq}, \quad k = 0, \ldots, N-1$$

$\rightarrow$ p equality constraints: $p = n_{eq} N$

m inequality constraints: $m = n_{ineq} N$

3. Solve $\bar{\mathcal{P}}$ using off-the-shelf solver

Intro to powered descent guidance:
"pinpoint rocket landing"



ϕ: cant angle (angle between thruster and vertical)

n: number of thrusters

I_{sp}: specific impulse of engine (efficiency of rocket engine)
higher I_{sp} = more efficient engine

problem: want to go from $x_{init} \rightarrow x_{goal}$ while burning as little fuel as possible