Lecture 7 2025-09-16

Last time: KKT conditions for inequality constrained optimization

Today: primal-dual interior point method for convex quadratic programs

$$\min_x \ \frac{1}{2} x^T P x + p^T x \qquad P \in S_+^n \qquad p \in \mathbb{R}^n$$

subj. to: $Ax = b$ $\qquad A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$

$Gx \leq h$ $\qquad G \in \mathbb{R}^{p \times n} \quad h \in \mathbb{R}^p$

introduce slack variables: $s \geq 0 \qquad s \in \mathbb{R}^p$

$$Gx \leq h \iff \begin{aligned} Gx + s &= h \\ s &\geq 0 \end{aligned}$$

Want to enforce $s \geq 0$ → without Lagrange multipliers

→ don't enforce $s \geq 0$ explicitly

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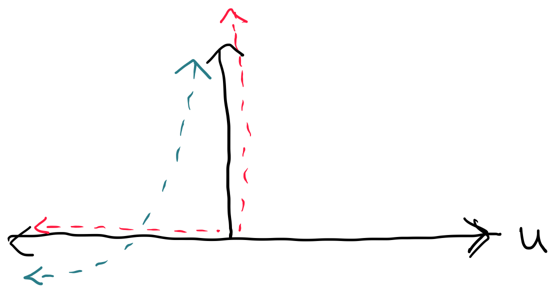
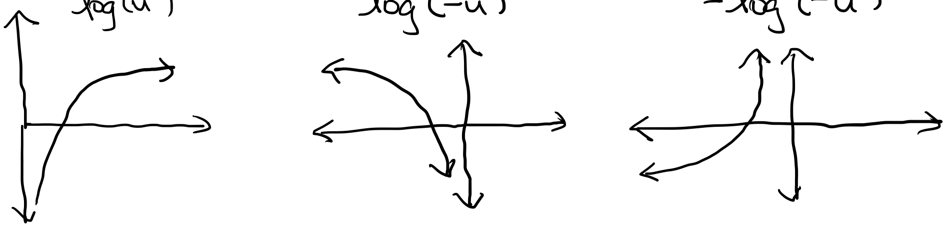
→ penalize violations to implicitly enforce $s \geq 0$

e.g. suppose we have some constraint $u \leq 0$

add penalty to violation: $I_-(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{otherwise} \end{cases}$



smooth approximation: $\Phi_\mu(u) = -\log(-u)$



→ if we have $s \geq 0$ $\quad s \in \mathbb{R}^p$

$$\rightarrow \Phi_\mu(-s) = -\mu \sum_{i=1}^{p} \log(s_i) \quad \text{where } \mu > 0$$

→ convert QP to barrier form:

$$\min_x \frac{1}{2} x^T P x + p^T x \qquad\qquad \min_x \frac{1}{2} x^T P x + p^T x + \Phi_\mu(-s)$$

$$\min_x$$

subj. to: $Ax = b$, $Gx \le h$ $\iff$ subj. to: $Ax = b$, $Gx + s = h$, $s \ge 0$

NOTE: as $\mu \to 0$, recover original solution

How to solve this barrier-form QP?

→ write Lagrangian and take Taylor expansion

$$\mathcal{L}(x, s, y, z) = \frac{1}{2}x^T P x + p^T x + \Phi_\mu(-s) + y^T(Ax - b) + z^T(Gx + s - h)$$

N.C.O:

(1) $\nabla_x \mathcal{L}(x^*, s^*, y^*, z^*) \quad = Px^* + p + A^T y^* + G^T z^*$

(2) $\nabla_y \mathcal{L}(x^*, s^*, y^*, z^*) = Ax^* - b$

(3) $\nabla_z \mathcal{L}(x^*, s^*, y^*, z^*) = Gx^* + s^* - h$

(4) $\nabla_s \mathcal{L}(x^*, s^*, y^*, z^*) = z^* - \mu s^{-1}$

$$\Phi_\mu(-s) = -\mu \sum_{i=1}^{p} \log(s_i)$$

$$\nabla_s \Phi_\mu(-s) = -\mu s^{-1} = \begin{pmatrix} -\mu/s_1 \\ \vdots \\ -\mu/s_p \end{pmatrix}$$

$$\rightarrow r(x, s, y, z) = \begin{pmatrix} Px + p + A^T y + G^T z \\ s \odot z - \mu \\ Gx + s - h \\ Ax - b \end{pmatrix}$$

Define: $D(s) = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_p \end{pmatrix}$ $\quad D(z) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_p \end{pmatrix}$

Use same approach as last time: given $(\bar{x}, \bar{s}, \bar{y}, \bar{z})$,

take Taylor series expansion and solve

for $(\Delta x, \Delta s, \Delta y, \Delta z)$ to find $r(x^*, s^*, y^*, z^*) = 0$

$$\rightarrow r(x, s, y, z) \approx r(\bar{x}, \bar{s}, \bar{y}, \bar{z}) + \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial s}\Delta s + \frac{\partial r}{\partial y}\Delta y + \frac{\partial r}{\partial z}$$

compute partials:

$$\frac{\partial r}{\partial x} = \begin{pmatrix} P \\ 0 \\ G \\ A \end{pmatrix}$$

$\frac{\partial r}{\partial s} = \Big(0 \quad$ $\Big(\bar{z}_1 \bar{s}_1 \quad$ $\Big(\bar{z}_1$

$$\frac{\partial r}{\partial s} = \begin{pmatrix} 0 \\ D(\bar{z}) \\ I \\ 0 \end{pmatrix} \leftarrow \bar{z} \odot \bar{s} = \begin{pmatrix} \bar{z}_1\bar{s}_1 \\ \vdots \\ \bar{z}_p\bar{s}_p \end{pmatrix} \rightarrow \frac{\partial r}{\partial s} = \begin{pmatrix} \bar{z}_1 & & \\ & \ddots & \\ & & \bar{z}_p \end{pmatrix} = D(\bar{z})$$

$$\frac{\partial r}{\partial z} = \begin{pmatrix} G^T \\ D(\bar{s}) \\ 0 \\ 0 \end{pmatrix} \qquad \frac{\partial r}{\partial y} = \begin{pmatrix} A^T \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\underbrace{\begin{pmatrix} P & 0 & G^T & A^T \\ 0 & D(\bar{z}) & D(\bar{s}) & 0 \\ G & I & 0 & 0 \\ A & 0 & 0 & 0 \end{pmatrix}}_{\text{KKT matrix}} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta z \\ \Delta y \end{pmatrix} = -\underbrace{\begin{pmatrix} P\bar{x} + p + A^T\bar{y} + G^T\bar{z} \\ \bar{s} \odot \bar{z} - \mu\vec{1} \\ G\bar{x} + \bar{s} - h \\ A\bar{x} - b \end{pmatrix}}_{\text{Residual vector}}}$$

(note minus sign in front)

NOTE: KKT matrix depends on $\bar{s}$ and $\bar{z}$ term, so this linear system must be resolved at each iteration

Primal-Dual Interior Point Method

1. Solve for (x^0, s^0, y^0, z^0) such that $s^0 \geq 0$ and $z^0 \geq 0$

for $k = 1 \to N_{max}$

2. Construct KKT matrix using $(x^{k-1}, s^{k-1}, y^{k-1}, z^{k-1})$
3. Compute residual vector using $(x^{k-1}, s^{k-1}, y^{k-1}, z^{k-1})$
4. Solve for affine step $(\Delta x^k_{aff}, \Delta s^k_{aff}, \Delta y^k_{aff}, \Delta z^k_{aff})$

5. Solve for centering-corrector step $(\Delta x_{cc}^k, \Delta s_{cc}^k, \Delta y_{cc}^k, \Delta z_{cc}^k)$

6. Line search $\alpha > 0$ such that $s^k \geq 0$ and $z^k \geq 0$

$$x^k = x^{k-1} + \alpha (\Delta x_{aff}^k + \Delta x_{cc}^k)$$

$$s^k = s^{k-1} + \alpha (\Delta s_{aff}^k + \Delta s_{cc}^k) \geq 0$$

$$y^k = y^{k-1} + \alpha (\Delta y_{aff}^k + \Delta y_{cc}^k)$$

$$z^k = z^{k-1} + \alpha (\Delta z_{aff}^k + \Delta z_{cc}^k) \geq 0$$

7. if $\| \text{residual}(x^k, s^k, y^k, z^k) \|_2 \leq \varepsilon_{tol}$

   break

To recap:

1. Applied the generic N.C.O. from last time to a convex QP

2. Introduced a barrier function to enforce inequality constraints "implicitly" at each iteration
3. Derived N.C.O. for the Lagrangian corresponding
   to barrier QP-reformulation
4. Defined a residual function for these N.C.O. and derived KKT system