Lecture 6 2025-09-11

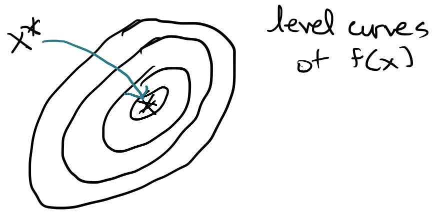
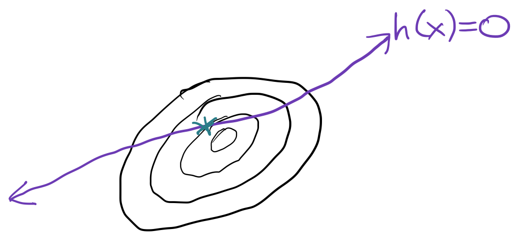
Last time: Newton's method with equality con.

Today: N.C.O. with inequality constraints

Review necessary conditions of optimality:

if x^* is an optimizer, then it must satisfy:

Unconstrained Equality Constrained

Unconstrained	Equality Constrained
	
$\min_x f(x)$ $\quad f \in \mathcal{C}^2$	$\min_x f(x)$ subj. to: $h(x)=0$ $\quad f, h \in \mathcal{C}^2$
N.C.O.: $\nabla_x f(x^*) = 0$	$\nabla_x f(x^*) + y_i^* \nabla_x h_i(x^*) = 0$

$h_i(x^*) = 0$

$\Downarrow$ succinctly

$\mathcal{L}(x,y) = f(x) + y_i h_i(x)$

$$r(x,y) \begin{cases} \nabla_x \mathcal{L}(x^*, y^*) = 0 \\ \nabla_y \mathcal{L}(x^*, y^*) = 0 \end{cases}$$

Recall: for equality constrained optimization, invert

KKT matrix: $\begin{matrix} H & A \\ A^T & 0 \end{matrix}$

1. A is full rank
2. Hessian is positive definite $\forall x$

e.g. quadratic program with equality

constraints

$$\min_{x} \ \frac{1}{2} x^T P x + p^T x$$

$$\text{subj. to: } \ Ax = b$$

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$ $\quad b \in \mathbb{R}^m$

$P \in S^n_+$ (i.e. P is a positive definite matrix of size $\mathbb{R}^{n \times n}$)

$\rightarrow$ this ensures $\nabla_{xx} f(x)$ is positive definite

$$\mathcal{L}(x,y) = \frac{1}{2} x^T P x + p^T x + y^T (Ax - b)$$

$$\rightarrow \quad \nabla_x \mathcal{L}(x,y) = Px + p + A^T y := 0$$

$$\nabla_y \mathcal{L}(x,y) = Ax - b := 0$$

given $x^{(0)}$ s.t. $Ax^{(0)} - b = 0$, can use feasible-start Newton method:

$$\nabla_x f(x) = Px + p \qquad \nabla_{xx} f(x) = P$$

$$\Rightarrow \begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ y \end{pmatrix} = - \begin{pmatrix} P x^{(k)} + p \\ 0 \end{pmatrix}$$

Optimization with linear inequality constraints

start with nonlinear form:

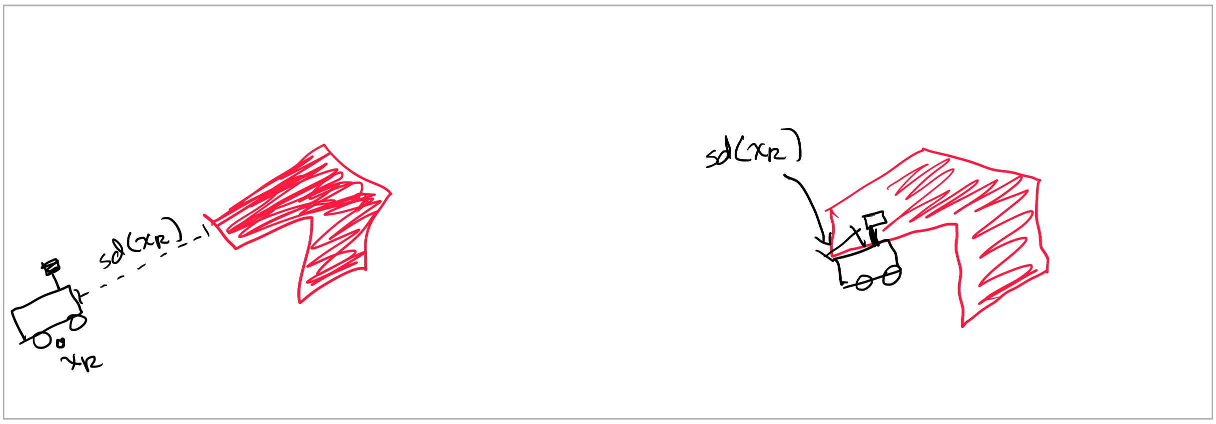
$$\min_x f(x)$$

subj. to: $g_i(x) \leq 0 \quad i = 1, \ldots, p$

e. g. practical examples $u_k \leq u_{max} \quad u \in \mathbb{R}^{n_u}$

$u_{min} \leq u_k \rightarrow u_{min} - u_k \leq 0$

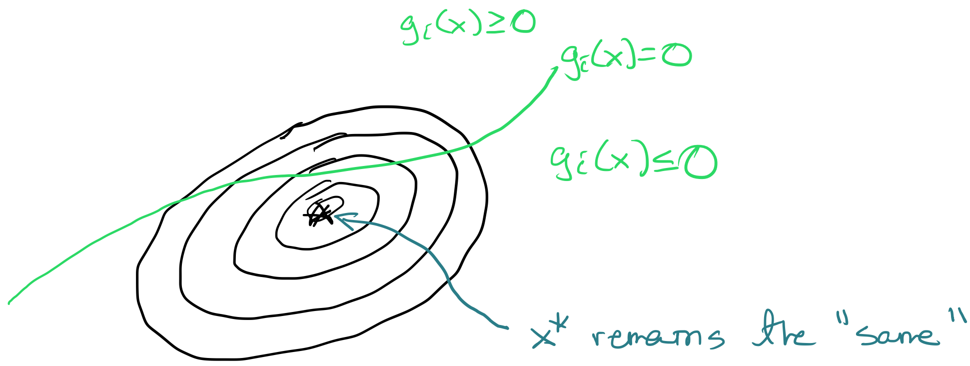
signed distance field: $sd(x_R) \geq r \rightarrow r - sd(x_R) \leq 0$



How do we get necessary conditions of optimality?

two cases:

1. $g_i(x) \leq 0$ doesn't affect x^*

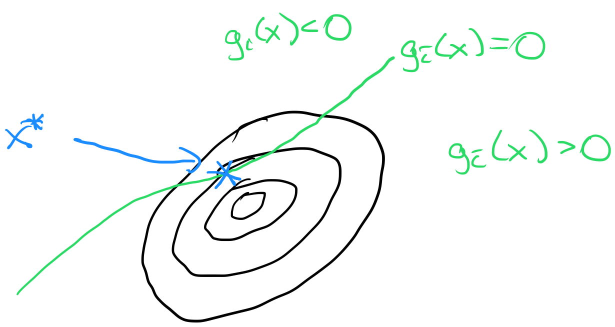


if $g_i(x^*) < 0$ (i.e. constraint is inactive)

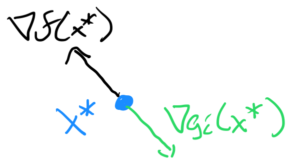
$\rightarrow \nabla_x f(x^*) = 0$

2. $g_i(x) \leq 0$ "influences" x^*

$\rightarrow g_i(x^*)$ is active



"similar" constraint as last time:



$$\nabla_x f(x^*) = -z_i \nabla_x g_i(x^*)$$

where $z_i > 0$

if $g_i(x^*) = 0$

$$\rightarrow \nabla_x f(x^*) + z_i \nabla_x g_i(x^*) = 0$$

$$z_i > 0$$

Putting this together:

if $g_i(x^*) = 0$ // constraint active

$$\nabla_x f(x^*) + z_i \nabla_x g_i(x^*) = 0$$

else // constraint inactive

$$\nabla_x f(x^*) = 0$$

use complementarity constraint to put this succintly:

(1) $z_i^* \cdot g_i(x^*) = 0$

(2) $\nabla f(x^*) + z_i \nabla g_i(x^*) = 0$

(3) $g_i(x^*) \leq 0$

This yields:

if $g_i(x^*) < 0$

to satisfy (1), $z_i^* = 0$

then plug into (2), $\nabla f(x^*) = 0$

else

$g_i(x^*) = 0$

$\nabla f(x^*) + z_i \nabla g(x^*) = 0$

$z_i > 0$

N. C.O. for inequality constrained optimization:

$$\mathcal{L}(x,y) = f(x) + \sum_{i=1}^{p} z_i g_i(x)$$

1. $\nabla_x \mathcal{L}(x^*, z^*) = \nabla_x f(x^*) + \sum_{i=1}^{p} z_i^* \nabla_x g_i(x^*) = 0$

2. $\nabla_{z_i} \mathcal{L}(x^*, z^*) = g_i(x^*) \leq 0 \qquad i = 1, \dots, p$

3. $z_i^* \cdot g_i(x^*) = 0 \qquad i = 1, \dots, p$

4. $z_i^* \geq 0 \qquad i = 1, \dots, p$

Add back m equality constraints:

$$\min_x f(x) \quad f, g, h \in \mathcal{C}^2$$

$$\text{subj. to:} \quad g_i(x) \leq 0 \quad i=1,\dots,p$$
$$h_i(x) = 0 \quad i=1,\dots,m$$

$$\mathcal{L}(x,y,z) = f(x) + \sum_{i=1}^{m} y_i h_i(x) + \sum_{i=1}^{p} z_i g_i(x)$$

1. $\nabla_x \mathcal{L}(x^*, y^*, z^*) = \nabla_x f(x) + \sum_{i=1}^{m} y_i^* \nabla_x h(x^*) + \sum_{i=1}^{p} z_i^* \nabla_x g_i(x^*) := 0$

2. $\nabla_{y_i} \mathcal{L}(x^*, y^*, z^*) = h_i(x^*) = 0 \qquad i=1,\dots,m$

3. $\nabla_{z_i} \mathcal{L}(x^*, y^*, z^*) = g_i(x^*) \leq 0 \qquad i=1,\dots,p$

4. $z_i^* \cdot g_i(x^*) = 0 \qquad i=1,\dots,p$

5 $z_i^* \geq 0 \qquad i=1 \quad p$

5. $z_i^* \geq 0$ $i=1,\dots,p$

y & z are often called dual variables

x is the primal variable

Quadratic program with equality + inequality constraints

$$\min_x \frac{1}{2} x^T P x + p^T x \quad P \in S_+^n \quad p \in \mathbb{R}^n$$

$$\text{subj. to: } Ax = b \quad A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m$$

$$Gx \leq h \quad G \in \mathbb{R}^{p \times n} \quad h \in \mathbb{R}^p$$

to solve this, we'll introduce slack variable $s \geq 0$:

$$Gx \leq h \iff \begin{array}{l} Gx + s = h \\ s \geq 0 \quad s \in \mathbb{R}^p \end{array}$$

use line searches to ensure $s \geq 0$ and $z \geq 0$

$$\mathcal{L}(x, y, z, s) = \frac{1}{2} x^T P x + p^T x + y^T (Ax - b) + z^T (Gx + s - h)$$