Lecture 11 2025-09-30

Last time: powered descent guidance

Today: planning over orientations

Complication: until now, we've been optimizing over "flat" spaces, i.e., $x \in \mathbb{R}^{n_x}$

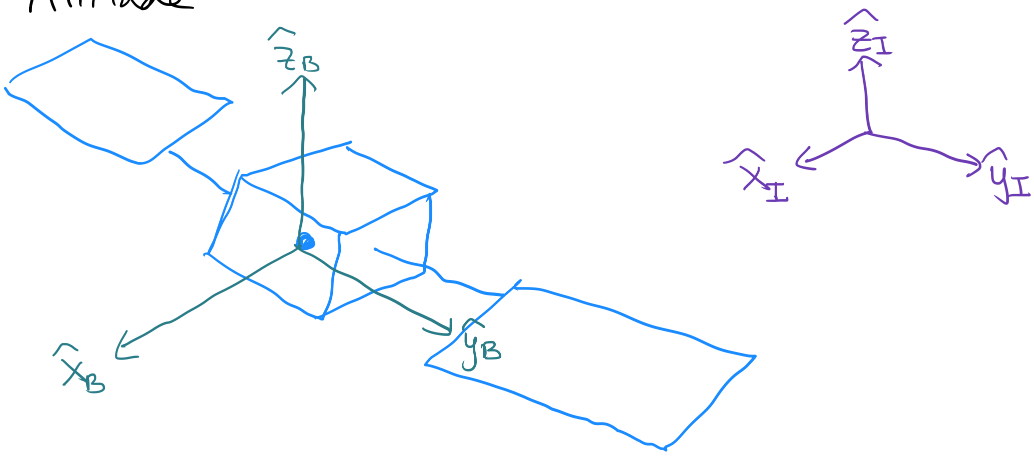
vector addition: $x_3 = x_1 + x_2$

Taylor series: $f(x) \approx f(\bar{x}) + \nabla_x f(\bar{x})^T \delta x$

→ this no longer holds when considering rotations

Attitude: the rotational orientation of a rigid body w.r.t. inertial frame

Attitude



Attitude: transform between $\vec{\mathcal{F}}_B$ and $\vec{\mathcal{F}}_I$

Attitude determination: what is C_{BI}, i.e., rotational transformation between $\mathcal{F}_B$ and $\mathcal{F}_I$?

Attitude control: controlling the spacecraft to yield some desired pointing vector

→ attitude determination & control system (ADCS)

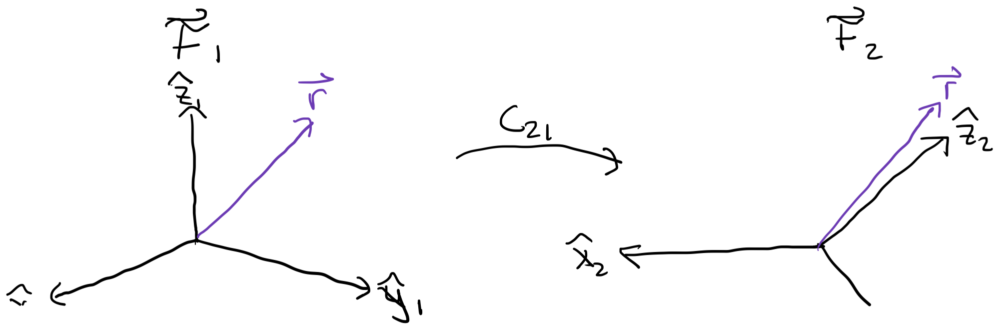
Optimal control for attitude planning:

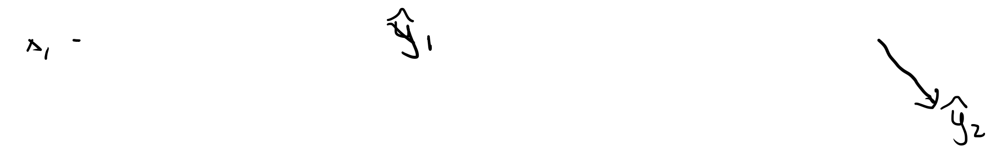
useful when there are challenging constraints and a simple feedback controller doesn't suffice

Will review three attitude parameterizations:

1. Direction cosine matrices
2. Euler angles
3. Quaternions

1. Direction cosine matrices (DCMs)





C_{21}: DCM that maps vectors resolved in $\vec{\mathcal{F}}_1$ to $\vec{\mathcal{F}}_2$

$$\vec{\mathcal{F}}_1 = \begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \\ \hat{z}_1 \end{pmatrix}$$

$\vec{\mathcal{F}}_1$ is an orthonormal basis

$$\rightarrow \quad \hat{x}_1 \cdot \hat{x}_1 = \hat{y}_1 \cdot \hat{y}_1 = \hat{z}_1 \cdot \hat{z}_1 = 1$$

$$\hat{x}_1 \cdot \hat{y}_1 = \hat{x}_1 \cdot \hat{z}_1 = \hat{y}_1 \cdot \hat{z}_1 = 0$$

$$\hat{z}_1 = \hat{x}_1 \times \hat{y}_1$$

$$\vec{r} = \vec{\mathcal{F}}_1^T \vec{r}_1 = \vec{\mathcal{F}}_2^T \vec{r}_2$$

where $\vec{r}_1$ is the coordinates of $\vec{r}$ in $\vec{\mathcal{F}}_1$

$$C_{21} = \begin{pmatrix} \hat{x}_2 \cdot \hat{x}_1 & \hat{x}_2 \cdot \hat{y}_1 & \hat{x}_2 \cdot \hat{z}_1 \\ \hat{y}_2 \cdot \hat{x}_1 & \hat{y}_2 \cdot \hat{y}_1 & \hat{y}_2 \cdot \hat{z}_1 \\ \hat{z}_2 \cdot \hat{x}_1 & \hat{z}_2 \cdot \hat{y}_1 & \hat{z}_2 \cdot \hat{z}_2 \end{pmatrix}$$

properties:

$$C_{21}^T C_{21} = I \iff C_{21}^{-1} = C_{21}^T$$

$$\det |C_{21}| = 1$$

$\Rightarrow$ special orthogonal group

$$SO(3) = \{ R \in \mathbb{R}^{3\times3} \mid R^T R = I \text{ and } \det|R| = 1 \}$$

"manifold"

successive rotations: $C_{31} = C_{32} C_{21}$

kinematics: $\dot{R} = R \omega^{\times}$

$$\omega^{\times} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$ "skew-symmetric matrix"

e.g., point spacecraft such that it ends up at R_f

$$\min_{R_{0:N}, \omega_{0:N}} \sum_{k=0}^{N} g(R_k, \omega_k)$$

subj. to: $R_0 = R_{init}$

$R_N = R_f$

$\dot{R}_k = R_k \omega_k^{\times}$

NOTE: active vs. passive rotation matrices

↳ today: passive DCMs

Downsides:

1. Rotations are inherently three degrees-of-freedom but DCMs have 9 parameters

2. Need to stay "on manifold"

$$R_k \in SO(3)$$

$$R_k^T R_k = I \text{ and } \det|R_k| = 1$$

→ these are highly nonlinear (non-convex)

→ these are highly nonlinear (non-convex) constraints

3. Kinematics need to preserve momentum

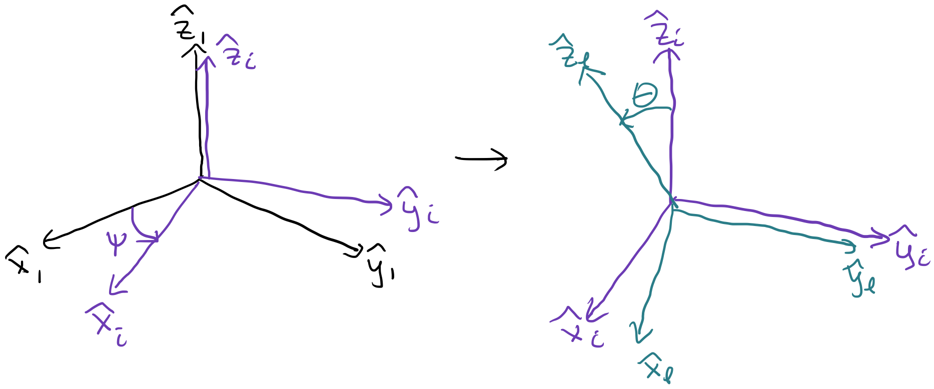
$$\dot{R} = R\omega^{\times} \rightarrow R_{k+1} = R_k + \Delta t\, R_k \omega_k^{\times}$$

⇒ Attitude is related to three degree-of-freedom motion, but DCMs are over parametrized

2. Euler angle

Today: C_{321}-sequence

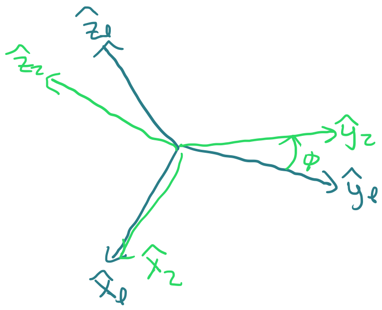
$\hat{z}$



First: yaw, rotate ψ about $\hat{z}_1$

Second: pitch, rotate θ about $\hat{y}_i$

Third: roll, rotate ϕ about $\hat{x}$



If $C_{321} = \underbrace{C_x(\phi)}_{\text{principal rotation about } \hat{x}} C_y(\theta)\, C_z(\psi)$

Two singularities can occur when $\theta = \pi/2$

1. $C_{321}(\phi, \pi/2, \psi) = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi - \psi) & \cos(\phi - \psi) & 0 \\ \cos(\phi - \psi) & -\sin(\phi - \psi) & 0 \end{pmatrix}$

cannot "distinguish" ϕ and ψ

cannot "distinguish" ϕ and ψ

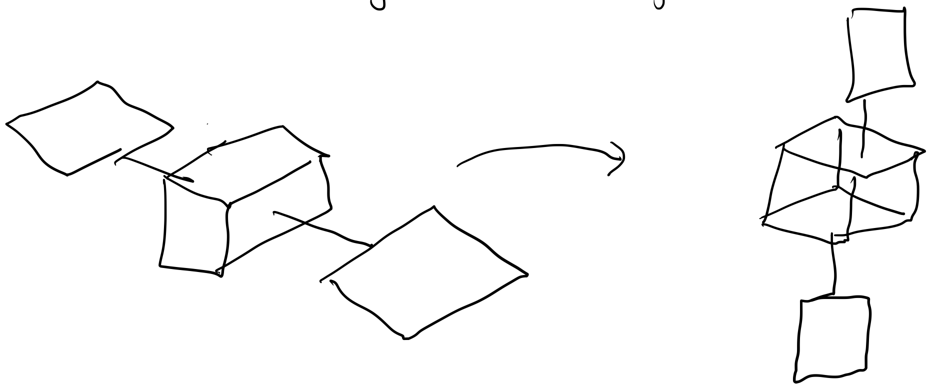
2. Kinematic singularity at $\theta = \pi/2$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

→ blows up at $\theta = \pi/2$

Pro: intuitive to think about

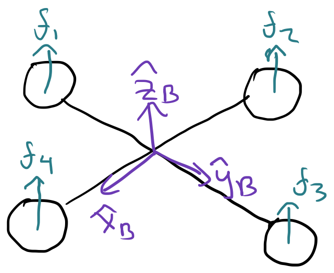
Cons: run into singularities for large rotations



For translations: $\left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\|_2$ is distance

For rotation: $\left\| \begin{pmatrix} \psi_1 \\ \phi_1 \\ \theta_1 \end{pmatrix} - \begin{pmatrix} \psi_2 \\ \phi_2 \\ \theta_2 \end{pmatrix} \right\|_2$ is not a "good" distance

Quadrotor trajectory generation:



Operate in 6-DoF

(i.e, translate and rotate)

but they are underactuated

→ Differentially flat system for quadrotors

↳ can characterize configuration of system using only flat outputs for system

→ D. Mellinger & V. Kumar, "Minimum Snap Trajectory Generation and Control for Quadrotors", ICRA 2011.

→ Showed that one can plan over flat output space $\sigma(t) = (x, y, z, \psi)$

→ given $(x(t), y(t), z(t), \psi(t))$, can recover

$\omega(t)$, $R(t)$, $u(t)$

$\omega(t), R(t), u(t), \ldots$

e.g., $$\min_{x(t), y(t), z(t), \psi(t)} \sum f(x, \dot{x}, y, \dot{y}, z, \dot{z}, \psi, \dot{\psi})$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\psi} \end{pmatrix} = (\text{double integrator})$$

Disadvantage: cannot enforce constraints for e.g., $\omega(t)$ without introducing nonlinearity

From last time: orientations are not "flat"

pitch $= \pi/2 \rightarrow$

$$C_{21}(\phi, \theta = \pi/2, \psi) = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi - \psi) & \cos(\phi - \psi) & 0 \\ \cos(\phi - \psi) & -\sin(\phi - \psi) & 0 \end{pmatrix}$$

In Euclidean spaces: $\|x - y\|_2$ is a valid distance measure

But with Euler angles

But with Euler angles,

e.g. $\phi = 10^\circ, \theta = 90^\circ, \psi = 20^\circ$

$\hookrightarrow |(\phi,\theta,\psi)-(0,0,0)|_{\ell_1} = 120^\circ$

$$C_{21} = \begin{pmatrix} 0 & 0 & -1 \\ \sin(-10^\circ) & \cos(-10^\circ) & 0 \\ \cos(-10^\circ) & -\sin(-10^\circ) & 0 \end{pmatrix}$$

e.g. $\phi = 0, \ \theta = 90^\circ, \ \psi = 10^\circ$

$\hookrightarrow |(\phi,\theta,\psi)-(0,0,0)|_{\ell_1} = 100^\circ$

$$C_{21} = \begin{pmatrix} 0 & 0 & -1 \\ \sin(-10^\circ) & \cos(-10^\circ) & 0 \\ \cos(-10^\circ) & -\sin(-10^\circ) & 0 \end{pmatrix}$$

Terminology: Lie group

A group $(\mathcal{G}, \circ)$ consists of a set $\mathcal{G}$ with a composition operator $\circ$ such that given $x, y, z \in \mathcal{G}$, the following hold:

1. Closure under $\circ$

$$x \circ y \in G$$

2. Identity $\mathcal{E}$,

$$\mathcal{E} \circ x = x \circ \mathcal{E} = x$$

3. Inverse x^{-1}:

$$x^{-1} \circ x = x \circ x^{-1} = \mathcal{E}$$

4. Associativity:

$$(x \circ y) \circ z = x \circ (y \circ z)$$

Lie group: a group whose elements exist on a smooth manifold

a space that locally looks like the Euclidean space

e.g. Rational numbers: a group but not a Lie group

$$Q = \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$$

1. Closure under $\cdot$:

$$\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2} := \frac{p_3}{q_3} \in Q$$

$\rightarrow$ Q has "holes", so not a part of a smooth manifold

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e.g. Euclidean space $\mathbb{R}^n$ is a Lie group

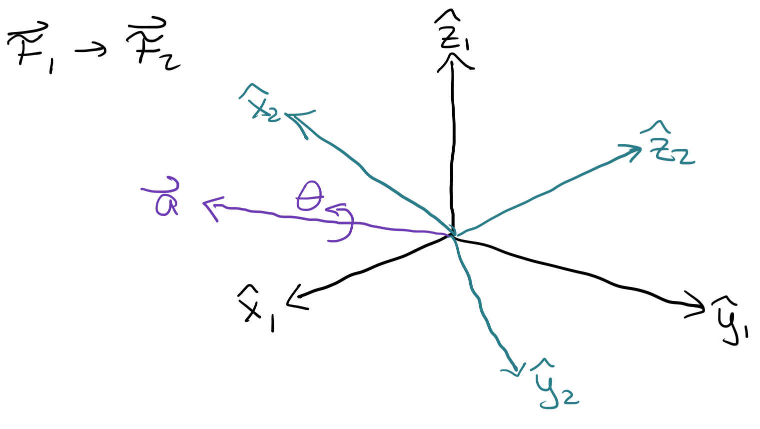
e.g. special orthogonal group:

$$SO(3) = \{ R \mid R \in \mathbb{R}^{3\times3}, R^T R = I, \det|R| = 1 \}$$

Quaternions

Euler's Theorem: the most general motion of a rigid body with one point fixed is a rotation about an axis through that point

→ axis-angle representations



$\hat{y}_2$

→ rotation described as rotating θ about $\vec{a}$

$$C_{21} = \cos\phi \, I + (1-\cos\phi)\vec{a}\vec{a}^T - \sin\theta \underbrace{\vec{a}^{\times}}_{\text{skew-symmetric matrix}}$$

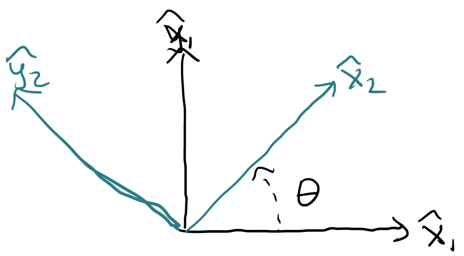
→ Use this to define a quaternion:

$$q = \begin{pmatrix} q_s \\ q_v \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, \vec{a} \end{pmatrix} \begin{matrix} \} \text{scalar component} \\ \} \text{vector component} \end{matrix}$$

→ have a singularity free and vector representation of the attitude

1. Quaternions are a "double cover"

e.g.



→ $q = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}$

→ this is the same as rotating $2\pi - \theta$ about $-\vec{a}$

$$\cos\left(\frac{2\pi-\theta}{2}\right) = \cos(\pi - \theta/2)$$
$$= \cos\pi \cos\theta/2 + \sin\pi \sin\theta/2$$
$$= -\cos\theta/2$$

$$\sin\left(\frac{2\pi-\theta}{2}\right) = \sin(\pi - \theta/2)$$
$$= \sin\pi \cos\theta/2 - \cos\pi \sin\theta/2$$

$$= \sin\pi \cos\theta/2 - \cos\pi \sin\theta/2$$
$$= \sin\theta/2$$

$$\rightarrow \begin{pmatrix} \cos\left(\frac{2\pi-\theta}{2}\right) \\ \sin\left(\frac{2\pi-\theta}{2}\right)\begin{pmatrix}0\\0\\-1\end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\cos\theta/2 \\ -\sin\theta/2 \begin{pmatrix}0\\0\\1\end{pmatrix} \end{pmatrix} = -q$$

in practice, "canonicalize" the quaternion

↳ set q such that $\cos(\theta/2) \geq 0$

2. $$q^T q = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, \vec{a} \end{pmatrix}^T \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, \vec{a} \end{pmatrix}$$

$$= \cos^2\theta/2 + \sin^2\theta/2\, \underbrace{\vec{a}^T\vec{a}}_{=1} = c^2\theta/2 + s^2\theta/2 = 1$$

$\rightarrow$ norm constraint: $\|q\|_2 = 1$

Lie group of quaternions:

$$S^3 = \{ q \in \mathbb{R}^4 \mid q^T q = 1 \}$$

e.g. $\dot{q} = f(q, \omega) \rightarrow \underbrace{q_{k+1}}_{\text{won't have unit norm}} = q_k + \Delta t\, f(q_k, \omega_k)$

Advantages

1. Vector representation
2. Singularity free

2. Singularity free

Disadvantage

1. "Double cover" of $SO(3)$

2. Unit norm constraint

Successive rotations:

$$C_{31} = C_{32} C_{21}$$

if $C_{21} = C(s_1, v_1)$ and $C_{32} = (s_2, v_2)$,

then $C_{32} = C(s_3, v_3)$ where

$$v_3 = v_1^{\times} v_2 + s_1 v_2 + s_2 v_1$$

$$s_3 = s_1 s_2 - v_1^T v_2$$

Kinematics: $\dot{q} = \frac{1}{2} \Omega(\omega) q$ $\qquad q = \begin{pmatrix} q_s \\ q_v \end{pmatrix}$ ← scalar ← vector

where $\underbrace{\Omega(\omega)}_{\text{angular velocity}} = \begin{pmatrix} 0 & -\omega^T \\ \omega & -\omega^{\times} \end{pmatrix}$

Trajectory optimization with quaternions:

Main challenges: 1. Deal with unit norm constraint

2. How to define distance metric

Different ways to write dynamics:

1. Enforce $\|q_k\|_2 = 1$ explicitly

$$\rightarrow q_{k+1} = q_k + \Delta t\, f(q_k, \omega_k)$$
$$\|q_k\|_2 = 1$$
$$\|q_{k+1}\|_2 = 1$$

In practice: unit norm constraints are very brittle

2. $$q_{k+1} = \frac{q_k + \Delta t\, f(q_k, \omega_k)}{\|q_k + \Delta t\, f(q_k, \omega_k)\|_2}$$

$q_k + \Delta t \, f(q_k, \omega_k)\|_2$

Works "slightly" better

3. Lie group variational integrator

↳ specialized integrator to ensure update equation stays "on manifold"

→ Preserve manifold constraint, but highly nonlinear

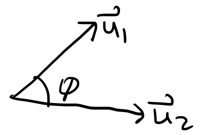
Cost functions:

On $\mathbb{R}^n$, $\|x-y\|_2$ is a valid distance metric

On S^3, q and $-q$ are same rotation

↳ $\| q - (-q) \|_2 = 2$

One possibility: given $\vec{u}_1$ and $\vec{u}_2$,



$\vec{u}_1 \cdot \vec{u}_2 = \|\vec{u}_1\|_2 \|\vec{u}_2\|_2 \cos\varphi$

↳ want $\varphi \to 0$

$\vec{q}_1 \cdot \vec{q}_2 = \|\vec{q}_1\|_2 \|\vec{q}_2\|_2 \cos\phi = \cos\phi$

want $\cos\phi \to 1$

since $\cos 0 = 1$

→ $d(q_1, q_2) = 1 - |\vec{q}_1^{\,T} \vec{q}_2|$

Optimal control formulation: want to start from $q(t_0) = q_{init}$ and drive system to $q(t_f) = q_{goal}$

$$\rightarrow \min_{q_{0:N},\, \omega_{0:N},\, T_{0:N}} \sum_{k=1}^{N} 1 - |q_k^T q_g|$$

subj. to: $q_{k+1} = \dfrac{q_k + \Delta t\, f(q_k, \omega_k)}{\| q_k + \Delta t\, f(q_k, \omega_k) \|_2}$

$J\dot{\omega} + \omega \times J\omega = T$

$$\omega_{k+1} = \omega_k + \Delta t\, J^{-1}(T_k - \omega_k \times J\omega_k)$$

$$\omega_{min} \leq \omega_k \leq \omega_{max}$$

$$T_{min} \leq T_k \leq T_{max}$$